

On minimal integer representations of weighted games

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Abstract

We study minimum integer representations for the weights of weighted games, which is linked with some solution concepts in game theory. Closing some gaps in the existing literature we prove that each weighted game with two types of voters admits a unique minimum integer presentation and give examples for more than two types of voters without a minimum integer representation. We characterize the possible weights in minimum integer representations and give examples for $t \geq 4$ types of voters without minimum integer representations preserving types.

Key words: weighted games, Minimum realizations, Realizations with minimum sum

JEL: C71, C65, D70

1. Introduction

Simple games, or positive switching functions, can be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo. Weighted voting games, or positive threshold functions, are possibly the most interesting subclass of simple games. Roughly speaking, in a weighted game a non-negative weight is assigned to each voter and a preset quota is specified, winning coalitions are those that can force a victory, i. e. the sum of their weights equal or surpass the preset quota. Weighted voting games naturally appear in several different contexts apart from voting, like reliability (see Ramamurthy [28]) or neural networks (see Elgot [7] or Freixas and Molinero [9]) among other technological fields.

The number of simple games on a fixed set $N = \{1, \dots, n\}$ is finite, of course, but it grows very rapidly with an increasing number of voters n since we are dealing with sets of sets. Indeed, every family of pairwise independent subsets of N can serve as the set of minimal winning coalitions defining a simple game. Two subsets are independent if neither contains the other. Families of independent subsets are sometimes called “Sperner families“, “coherent systems“, or “clutters“, and their enumeration and classification have occupied mathematicians since Dedekind in the 19th century. In his 1897 work he determined the exact number of simple games with four or fewer players. Since that time simple games have been investigated in a variety of different mathematical contexts. An account of some these works will be found in: Sperner [30], Isbell [15], Golomb [13], Muroga et al. [24, 25], Hoeffding [14], Shapley and Shubik [29], Dubey and Shapley

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[4], Kurz and Tautenhahn [20, 21, 31], Freixas and Molinero [10, 11], Krohn and Sudhölter [19], Keijzer et al. [17, 18]. Although the number of weighted games compared with simple games is small, it grows very rapidly and there are no enumerations for more than nine voters.

Integer representations are very common in practice and minimum integer representations, if they exist, constitute the most efficient way to represent weighted games. Geometrically, the set of equivalent integer representations of a weighted game depicts an unbounded cone with either a vertex or without it. Hence, a natural question arises. For which weighted games exists a minimum weighted representation? Or, in other words, for which weighted games its associated integer cone has a vertex? Symmetric games, i. e. games where all players have an equivalent role in the game and therefore characterized by one single type of equivalent voters, admit a unique minimum integer representation. But, it is known that it does not always exist a unique minimal representation in integers for a weighted game. Muroga et al. [24] in their exhaustive enumeration of threshold functions (or, equivalently, weighted games) uncovered several cases with as few as eight players in which two symmetric players must be given different weights in a minimal integer representation; e. g. $[12; 7, 6, 6, 4, 4, 4, 3, 2] = [12; 7, 6, 6, 4, 4, 2, 3]$. Moreover, they verified that all weighted games with less than eight players admit a minimum integer representation.

Their example has four types of players (a type here is an element of a partition of N formed by equivalent voters), and each type contains players with the same weights excepting for the last type, which contains players with weights 3 and 2. To our knowledge it is not known whether there exist weighted games without a unique representation in integers with less than either two or three types of players. The main goal of this paper is to ascertain what occurs for these two cases, filling the existing gap in the theory of weighted games. Previously to the Muroga et al.'s example, Isbell [15] had exhibited a remarkable 12-player example in which the affected players are not symmetric. Thus, even if we additionally require that all players of equal type have equal weights, the existence of a unique minimum integer representation preserving types is not necessarily implied. Similar questions emerge to be important in this more restrictive context. Freixas and Molinero [10, 11] uncovered several cases with as few as 9 players and checked the nonexistence of such examples for less than 9 players, see also [20]. All the examples they listed have at least 5 types of players. It concerns us to ascertain what occurs for less than 5 types. We would like to remark that homogeneous games admit a unique minimal representation as shown by Ostmann [26].

A natural third purpose emerges to be significant, whenever it does not exist a minimum integer representation for a weighted game in either of the two above described situations. In these situations, at least two integer representations become minimal, but is it possible to generate weighted games with more than two minimal representations? And, further, is it possible to construct a weighted game with some established number of minimal integer representations? As far as we know, all the examples showed until now without minimum integer representations (either preserving types or not) have only two minimal elements. Additional results we introduce here concern: bounds on the number of non-isomorphic weighted games depending on the number of voters and on the number of types of voters, and the existence of a weighted game in minimum integer representation for any pair of two coprime integer weights.

Minimum integer representations of weighted games are important in game theory: Peleg [27] proved that for homogeneous weighted decisive games the nucleolus (a well-known solution concept in game theory) coincides with the minimum integer representation preserving types. Also in the cases where there are several minimum sum integer representations preserving types there are connections linking minimal integer realizations with the least core (another solution concept in game theory) and the nucleolus of weighted decisive games [19].

The paper is organized as follows. In Section 2 we precisely define the classes of complete simple games and weighted games. For complete simple games we state a parameterization theorem by Carreras and Freixas in Subsection 2.1, which completely characterizes these objects up to isomorphism using linear

inequalities. The subclass of weighted games can be defined via the non-emptiness of certain polytopes as outlined in Subsection 2.2. The precise details of minimum integer representations are stated in Subsection 2.3. In Section 3 we present constructions for weighted games without a unique minimum integer representation for small t (Subsection 3.1) or with those with more than two minimum integer representations (Subsection 3.2). In Subsection 3.3 we study the question which weights may occur in a minimum integer representation. Our main theorem, that each weighted game with two types of voters admits a unique minimum integer representation, is given in Section 4. Some extra information on the enumeration or bounds on the number of weighted games which emerge from our previous results or complete those results are stated in Section 5. We end with a conclusion in Section 6.

2. Weighted voting games and complete simple games

So far we have informally defined weighted games via their weights w_i and their quota q . As mentioned in the introduction there are several representations for the same weighted game, e. g. $[3; 2, 1, 1, 1]$, $[4; 3, 2, 1, 1]$, $[11; 9, 5, 5, 4]$, $[q; q-1, x, x, x]$ and $[q; q-2, x, x, x]$ with $q \geq 6$ and $\lceil \frac{q}{3} \rceil \leq x \leq \lfloor \frac{q-1}{2} \rfloor$ all represent the same weighted game because the subsets of N whose weights equal or surpass the quota are invariant for all of them. So it makes sense to have a closer look at the underlying discrete structure. From a very general point of view one can describe voting methods using Boolean functions $\chi : 2^N = \{U \mid U \subseteq N\} \rightarrow \{0, 1\}$, where we may replace the subsets of N (called coalitions in the voting setting) by its characteristic function to obtain a genuine Boolean function. $\chi(U) = 1$ has the meaning that coalition U is winning. Assuming $\chi(\emptyset) = 0$, $\chi(N) = 1$, and $\chi(U \setminus \{u\}) \leq \chi(U)$ for all $u \in U$ leads to **simple games** (positive switching functions, positive or monotone Boolean functions). A simple game will be denoted from now on by (N, χ) . A well studied subclass of simple games (and superclass of weighted games) arises from Isbell's desirability relation [16]: We write $i \sqsupseteq j$ for two voters $i, j \in N$ iff we have $\chi(\{i\} \cup U \setminus \{j\}) \geq \chi(U)$ for all $j \in U \subseteq N \setminus \{i\}$ with the meaning that voter i is at least as influential as voter j . A pair (N, χ) is called **complete simple game** if it is a simple game and the binary relation \sqsupseteq is a total preorder. We abbreviate $i \sqsupseteq j, j \sqsupseteq i$ by $i \sqsubseteq j$ and assume $1 \sqsupseteq 2 \sqsupseteq \dots \sqsupseteq n$. For an extensive overview on voting methods we refer the interested reader to [32].

Whenever $i \sqsubseteq j$ voter i is as influential (strong) in the game as voter j , meaning that it does not matter which one of both takes part in a coalition. So we can partition the whole set N of voters into equivalence classes N_1, \dots, N_t and say that the complete simple game consists of t types of voters. By n_i we denote the cardinality of the set N_i for $1 \leq i \leq t$. The *type* of a coalition is formed by a set of coalitions and can be described by a vector (m_1, \dots, m_t) meaning m_i -out-of- n_i voters (from the set N_i) for $1 \leq i \leq t$. We can transfer the monotonic order \leq via

$$\chi((m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_t)) \leq \chi((m_1, \dots, m_t))$$

for $1 \leq i \leq t$ and the remaining part of the partial order \leq via

$$\chi((m_1, \dots, m_{i-1}, m_i - 1, m_{i+1} + 1, m_{i+2}, \dots, m_t)) \leq \chi((m_1, \dots, m_t))$$

for $1 \leq i \leq t-1$. We call the minimal vectors in this poset which are winning **shift-minimal winning coalitions** and the maximal vectors which are losing **shift-maximal losing coalitions**. A complete simple game is uniquely characterized by the set \mathcal{W}^s of its shift-minimal winning coalitions or the set \mathcal{L}^s of its shift-maximal losing coalitions. W.l.o.g. we mainly deal with types of coalitions, i. e. sets of coalitions, instead of coalitions and for this reason also call the vector (m_1, m_2, \dots, m_t) coalition.

As an example we consider the weighted game $[4; 3, 2, 1, 1]$ (which is the same as $[3; 2, 1, 1, 1]$), where we have $1 \sqsubset 2 \sqsubset 3 \sqsubset 4$ for the voters, i. e. $n_1 = 1$ and $n_2 = 3$. The shift-minimal winning coalitions are given by $(1, 1)$, $(0, 3)$ and the shift-maximal losing coalitions are given by $(1, 0)$, $(0, 2)$. Since $(1, 2) \succeq (1, 1)$ the coalition type $(1, 2)$ is also winning and $(0, 2)$ is losing due to $(0, 2) \preceq (0, 3)$.

2.1. A parameterization theorem for complete simple games

Carreras and Freixas have given an entire and non-redundant parameterization of complete simple games in [3]. Therefore by $\tilde{m}_i \bowtie \tilde{m}_j$ we denote the situation where neither $\tilde{m}_i \preceq \tilde{m}_j$ nor $\tilde{m}_i \succeq \tilde{m}_j$ holds, i. e. \tilde{m}_i, \tilde{m}_j are non-comparable coalitions. Additionally we denote the (decreasing) lexicographic order by \succ .

Theorem 2.1.

(a) Given are a vector

$$\tilde{n} = (n_1 \quad \dots \quad n_t) \in \mathbb{N}_{>0}^t$$

and a matrix

$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,t} \\ m_{2,1} & m_{2,2} & \dots & m_{2,t} \\ \vdots & \ddots & \ddots & \vdots \\ m_{r,1} & m_{r,2} & \dots & m_{r,t} \end{pmatrix} = \begin{pmatrix} \tilde{m}_1 \\ \tilde{m}_2 \\ \vdots \\ \tilde{m}_r \end{pmatrix}$$

satisfying the following properties

- (i) $m_{1,1} > 0$, $0 \leq m_{i,j} \leq n_j$, $m_{i,j} \in \mathbb{N}$ for $1 \leq i \leq r$, $1 \leq j \leq t$,
- (ii) $\tilde{m}_i \bowtie \tilde{m}_j$ for all $1 \leq i < j \leq r$,
- (iii) for each $1 \leq j < t$ there is at least one row-index i such that $m_{i,j} > 0$, $m_{i,j+1} < n_{j+1}$, and
- (iv) $\tilde{m}_i \succ \tilde{m}_{i+1}$ for $1 \leq i < r$.

Then, there exists a complete simple game (N, χ) associated to (\tilde{n}, \mathcal{M}) .

(b) Two complete games $(\tilde{n}_1, \mathcal{M}_1)$ and $(\tilde{n}_2, \mathcal{M}_2)$ are isomorphic if and only if $\tilde{n}_1 = \tilde{n}_2$ and $\mathcal{M}_1 = \mathcal{M}_2$.

In such a vector/matrix representation of a complete simple game the number of voters n is determined by $n = \sum_{i=1}^t n_i$. Although Theorem 2.1 looks technical at first glance, the necessity of the required properties can be explained easily. Obviously, $n_j \geq 1$, $m_{1,1} > 0$, and $0 \leq m_{i,j} \leq n_j$ must hold for $1 \leq i \leq r$, $1 \leq j \leq t$. If $\tilde{m}_i \preceq \tilde{m}_j$ or $\tilde{m}_i \succeq \tilde{m}_j$ then we would have $\tilde{m}_i = \tilde{m}_j$ or either \tilde{m}_i or \tilde{m}_j cannot be a shift-minimal winning coalition. If for a column-index $1 \leq j < t$ we have $m_{i,j} = 0$ or $m_{i,j+1} = n_{j+1}$ for all $1 \leq i \leq r$, then check that $g \sqsubset h$ for all $g \in N_j$, $h \in N_{j+1}$, which is a contradiction to the definition of the classes N_j and therefore also for the numbers n_j . Obviously a complete simple game does not change if two rows of the matrix \mathcal{M} are interchanged. Thus we must require an arbitrary ordering of the rows to avoid duplicities.

We would like to remark that for $t = 1$ only $r = 1$ is possible and the requirements reduce to $1 \leq m_{1,1} \leq n_1 = n$. Also for $t = 2$ one can easily give a more compact formulation for the requirements in

Theorem 2.1. A complete description of the possible values $n_1, n_2, m_{1,1}, m_{1,2}$ corresponding to a complete simple game with parameters $n, t = 2$, and $r = 1$ is given by

$$\begin{aligned} 1 &\leq n_1 \leq n - 1, \\ n_1 + n_2 &= n, \\ 1 &\leq m_{1,1} \leq n_1, \\ 0 &\leq m_{1,2} \leq n_2 - 1. \end{aligned} \tag{1}$$

For $t = 2$ and $r \geq 2$ such a complete and compact description is given by

$$\begin{aligned} 1 &\leq n_1 \leq n - 1, \\ n_1 + n_2 &= n, \\ m_{i,1} &\geq m_{i+1,1} + 1 & \forall 1 \leq i \leq r - 1, \\ m_{i,1} + m_{i,2} + 1 &\leq m_{i+1,1} + m_{i+1,2} & \forall 1 \leq i \leq r - 1. \end{aligned} \tag{2}$$

2.2. Recognizing weighted games

To decide whether complete simple games are weighted, we can utilize linear programs, see [32] for an overview on other methods. A complete simple game is weighted *if and only if* the following system of linear inequalities is feasible:

$$\begin{aligned} x^T w &\geq y^T w + 1 & \forall x \in \mathcal{W}^s, y \in \mathcal{L}^s \\ w_i &\geq w_{i+1} + 1 & \forall 1 \leq i \leq t - 1 \\ w_t &\geq 0 \end{aligned}$$

By introducing another variable q for the quota the number of inequalities can be reduced:

$$\begin{aligned} x^T w &\geq q & \forall x \in \mathcal{W}^s \\ y^T w &\leq q - 1 & \forall y \in \mathcal{L}^s \\ w_i &\geq w_{i+1} + 1 & \forall 1 \leq i \leq t - 1 \\ w_t &\geq 0 \end{aligned}$$

Since no integer variables are used so far these two linear programming formulations can be checked for feasibility in polynomial time. If the sum of weights is minimized the weights are integral or have very small denominators (at least for $n \leq 9$) nevertheless, see [20]. So one can obtain *small* integer representations very quickly. Introducing integer variables changes the linear programs to integer linear programs, whose solution is *NP*-hard in general. To our knowledge there is no known polynomial time algorithm to determine minimum sum integer representations. For some algebraic techniques we refer the interested reader to [2]. We would like to remark that those complete simple games which are not weighted can be represented as a finite intersection or union of weighted voting games, a construction which is also used in practice [8].

2.3. Minimum integer representations

An integer realization $[q; w_1, \dots, w_n]$ is called **minimum integer representation** if for all integer realizations $[q'; w'_1, \dots, w'_n]$ of the same game we have $w_i \leq w'_i$ for all $1 \leq i \leq n$. If we restrict the allowed representations to those where the voters of the same equivalence class N_i have an equal weight, we speak of **minimum integer representations preserving types**. In general, both representations need not exist

and indeed in this paper we study conditions where they exist and give examples where they do not exist. An integer realization $[q; w_1, \dots, w_n]$ is called **minimum sum representation** if we have $w(N) \leq w'(N)$ for all integer realizations $[q'; w'_1, \dots, w'_n]$ of the same game, where $w(U)$ is additively defined on the elements of U . If the integer realizations are restricted as before, we speak of **minimum sum representations preserving types**.

As mentioned before, minimum integer representations need not to exist and we will prove that they exist for some subclasses of weighted games or give examples of weighted games without minimum integer representations in the following subsections. Therefore we refrain that an integer representation $[q; w_1, \dots, w_n]$ represents a certain weighted game *if and only if* we have

$$\sum_{i \in U} w_i \geq q \quad \text{and} \quad \sum_{i \in U'} w_i \leq q - 1.$$

for each winning coalition U and each losing coalition U' .

We would like to note that the existence of a minimum integer representation for a weighted game implies the existence of a minimum integer representation preserving types, but the converse is not true. The example $[12; 7, 6, 6, 4, 4, 4, 3, 2] = [12; 7, 6, 6, 4, 4, 4, 2, 3]$ from the introduction has $[14; 8, 7, 7, 5, 5, 5, 3, 3]$ as a minimum representation preserving types.

3. Generating conspicuous examples of games without a minimum integer representation

Having in mind that one knows the existence of weighted games without a unique minimum integer representation for all $t > 3$, we are concerned in this section with this problem in the special case of $t = 3$ types of voters. As we shall see below, we propose a procedure to generate weighted games with three types of voters without a minimum integer representation based on the famous Coin-Exchange Problem of Frobenius [1].

Similarly, the existence of weighted games without a unique minimum integer representation preserving types is known for all $t > 4$. Thus the case $t = 4$ is under study here and we also propose a procedure to generate weighted games with four types of voters without a minimum integer representation preserving types.

Another objective of this section is to generate examples of weighted games with more than two minimal integer representations.

The last goal concerns weighted games with a unique minimum integer representation of coprime weights.

3.1. Weighted voting games without a unique minimum integer representation for small t

For $t \geq 3$ equivalence classes of voters there are examples of weighted games without a unique minimum integer representation and for $t \geq 4$ there are examples without a unique minimum integer representation preserving types. Here we present some new constructions for such weighted games based on the famous Coin-Exchange Problem of Frobenius [1]: Consider $n \geq 2$ integers $0 < a_1 < \dots < a_n$ with $\gcd(a_1, \dots, a_n) = 1$ as denominations of n different coins. The amount of money N that can be represented by using arbitrary multiples x_i of the given coins is given by $N = \sum_{i=1}^n a_i x_i$, where the x_i are non-negative integers.

If $a_1 > 1$ some amounts N can not be represented. The largest such N for a given problem is called the Frobenius number $g(a_1, \dots, a_n)$. Well known results in this context are $g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$ and that exactly $\frac{1}{2}(N + 1) = \frac{1}{2}(a_1 - 1)(a_2 - 1)$ non-negative integers are not representable for $\gcd(a_1, a_2) = 1$.

For our first construction we choose two coprime integers $b > a \geq 1$ as weights w_2 and w_3 , see Theorem 3.3, and integers q, w_1 such that $q - w_1$ is not representable using a, b , whereas $q - 2w_1 + 1$ is, and we additionally have $w_1 - 1 > b$. For $\tilde{n} = \begin{pmatrix} 2 & a & b \end{pmatrix}$ we consider the weighted games $[q; w_1 - \frac{1}{2}, w_2, w_3]$. Choosing $b = 7, a = 5, q = 35$, and $w_1 = 12$ yields the following example for

$t = 3$ types of voters: $\tilde{n} = \begin{pmatrix} 2 & 5 & 7 \end{pmatrix}$, $\mathcal{M} = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 5 & 3 & 0 \\ 1 & 3 & 2 & 4 & 5 & 0 & 3 & 7 \end{pmatrix}^T$. The matrix of the shift-

maximal losing coalitions is given by $\begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 & 4 & 2 \\ 0 & 2 & 0 & 3 & 1 & 4 \end{pmatrix}^T$. Minimizing w_1, w_2, w_3 , or the quota q

yields the solution $(35; 11.5, 7, 5)$. In this example the minimum integer sum representation is given by $(35; 12, 11, 7, 7, 7, 7, 7, 5, 5, 5, 5, 5, 5, 5)$, where the first two weights may be swapped.

For $t = 4$ types of voters we base our construction on the fact that there are no non-negative integers fulfilling $7u + 11v \in \{52, 59\}$ but that there are integers fulfilling $7u + 11v = 52 + 59 - 7 \cdot 11 + 1$ and consider the following example: $\tilde{n} = \begin{pmatrix} 1 & 1 & 7 & 11 \end{pmatrix}$,

$\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 3 & 1 & 0 & 5 & 3 & 1 & 0 & 7 & 6 & 4 & 2 & 0 \\ 2 & 5 & 0 & 3 & 6 & 8 & 1 & 4 & 7 & 9 & 0 & 2 & 5 & 8 & 11 \end{pmatrix}^T$. The matrix of the shift-maximal los-

ing coalitions is given by $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 4 & 2 & 0 & 5 & 4 & 2 & 0 & 6 & 5 & 3 & 1 & 1 & 1 \\ 0 & 3 & 1 & 4 & 7 & 0 & 2 & 5 & 8 & 1 & 3 & 6 & 9 & 11 & 11 \end{pmatrix}^T$. Minimizing $w_1, w_1 + w_2 +$

$11w_3 + 7w_4$, or $w_1, w_1 + w_2 + 11w_3 + 7w_4 + q$ yields the solution $(77; 24, 18, 11, 7)$. Minimizing w_2 yields the solution $(77; 25, 17, 11, 7)$ and minimizing w_3, w_4 , or q yields the solution $(77; 25, 18, 11, 7)$. Thus both $(77; 25, 17, 11, 11, 11, 11, 11, 11, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$ and $(77; 24, 18, 11, 11, 11, 11, 11, 11, 11, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$ are minimum sum representations preserving types.

3.2. Weighted voting games with more than two minimal integer representations

It would be nice to have an example of a weighted game with more than two different minimum sum representations preserving types. Our first idea was to choose $t = 4$, two coprime integers w_3, w_4 , and $\tilde{n} = \begin{pmatrix} 1 & 1 & w_4 & w_3 \end{pmatrix}$. We call an integer k representable if there are non-negative integers u, v fulfilling $uw_3 + vw_4 = k$. If there would exist integers $0 < l_2 < l_1 - 1 < w_3w_4$ such that $l_1, l_1 + 1, l_2, l_2 + 1$ are not representable but $w_3w_4 - l_1 - l_2 + 2$ is representable, then we would obtain such an example with $w_1 \geq w_3w_4 - l_1 - 2, w_2 \geq w_3w_4 - l_2 - 2$, and $w_1 + w_2 \geq 2w_3w_4 - l_1 - l_2 - 2$. Unfortunately there is Popoviciu's theorem which has the following consequence: For an integer $k \in [1, w_3w_4]$ not being divisible by w_3 or w_4 , exactly one of the numbers k and $w_3w_4 - k$ is representable. So there are u_1, u_2, v_1, v_2 with $w_3w_4 - l_1 - 1 = u_1w_3 + v_1w_4$ and $w_3w_4 - l_2 - 1 = u_2w_3 + v_2w_4$. Since $2w_3w_4 - (l_1 + 1) - (l_2 + 1) = (u_1 + u_2)w_3 + (v_1 + v_2)w_4$ the number $w_3w_4 - l_1 - l_2 + 2$ is not representable. For $t = 5$ we have another construction which works:

Proposition 3.1. *Let a, b be two coprime positive integers with $a > b$. We call an integer k representable if there exist non-negative integers u, v with $ua + vb = k$. Suppose we have integers $l_1 < l_2 < l_3$ fulfilling*

- (1) $a < \widehat{w}_i = ab - l_i - 1 < ab - 1$, l_i not representable, $l_i + 1$ representable for $1 \leq i \leq 3$,
- (2) $0 < l_1 + l_2 + l_3 - l_i - ab < ab$ is not representable for $1 \leq i \leq 3$, and
- (3) $0 < l_1 + l_2 + l_3 + 1 - 2ab < ab$ is representable.

The weighted game $(ab; \widehat{w}_1, \widehat{w}_2 + 1, \widehat{w}_3 + 1, a, \dots, a, b, \dots, b)$, $\tilde{n} = (1 \ 1 \ 1 \ b \ a)$ has the following three minimum sum representations preserving types:

- (1) $(ab; \widehat{w}_1, \widehat{w}_2 + 1, \widehat{w}_3 + 1, a, \dots, a, b, \dots, b)$
- (2) $(ab; \widehat{w}_1 + 1, \widehat{w}_2, \widehat{w}_3 + 1, a, \dots, a, b, \dots, b)$
- (3) $(ab; \widehat{w}_1 + 1, \widehat{w}_2 + 1, \widehat{w}_3, a, \dots, a, b, \dots, b)$

PROOF. Let w_1, w_2, w_3, w_4, w_5 be valid weights with quota q . From the corresponding inequality system we can deduce $q \geq ab$, $w_4 \geq \frac{q}{b} \geq a$, and $w_5 \geq \frac{q}{a} \geq b$. For $b > 3$ we can assume equality in a minimum sum representation. Since $l_i + 1$ is representable there exist non-negative integers with $u_i a + v_i b = l_i + 1 = ab - \widehat{w}_i$ for all $1 \leq i \leq 3$. Due to $\widehat{w}_1 + \underbrace{(ab - \widehat{w}_1)}_{=l_1+1} \geq ab$ we have $w_1 + u_1 w_4 + v_1 w_5 \geq q$ which is equivalent to

$w_1 \geq \widehat{w}_1$. If l_1 would be representable we would have $w_1 + l_1 \leq q - 1$ which is equivalent to $w_1 \leq \widehat{w}_1$. For $i = 2, 3$ we have $w_i + l_i \geq q$ and $w_i + l_i - 1 \leq q - 1$. The later inequality is equivalent to $w_i \leq \widehat{w}_i + 1$. The first inequality would be equivalent to $w_i \geq \widehat{w}_i + 1$. But since l_i is not representable we have only $w_i + l_i + 1 \geq q$ which is equivalent to $w_i \geq \widehat{w}_i$.

For $i = 2, 3$ we do not have $w_1 + w_i \leq \widehat{w}_1 + \widehat{w}_i + 1$ since $l_1 + l_i - ab$ is not representable but may have $w_1 + w_i \geq \widehat{w}_1 + \widehat{w}_i + 1$. Since $l_2 + l_3 - ab$ is not representable we do not have $w_2 + w_3 \geq \widehat{w}_2 + \widehat{w}_3 + 2$ but may have $w_2 + w_3 \leq \widehat{w}_2 + \widehat{w}_3 + 2$.

Since $l_1 + l_2 + l_3 + 1 - 2ab$ is representable and $\widehat{w}_1 + \widehat{w}_2 + 1 + \widehat{w}_3 + 1 + \underbrace{(ab - \widehat{w}_1 - \widehat{w}_2 - \widehat{w}_3 - 2)}_{=l_1+l_2+l_3-ab+1} \geq ab$ we have $w_1 + w_2 + w_3 + (ab - \widehat{w}_1 - \widehat{w}_2 - \widehat{w}_3 - 2) \geq q$, so that $w_1 + w_2 + w_3 \geq \widehat{w}_1 + \widehat{w}_2 + \widehat{w}_3 + 2$. \square

An example where the requirements of the previous proposition are fulfilled is given by $a = 17$, $b = 13$, $l_1 = 157$, $l_2 = 161$, $l_3 = 174$, $\widehat{w}_1 = 63$, $\widehat{w}_2 = 59$, and $\widehat{w}_3 = 46$.

A smaller example is given by $a = 13$, $b = 11$, $l_1 = 93$, $l_2 = 97$, $l_3 = 106$, $\widehat{w}_1 = 49$, $\widehat{w}_2 = 45$, and $\widehat{w}_3 = 36$.

Furthermore we have the following straightforward generalization:

Proposition 3.2. *Let a, b be two coprime positive integers with $a > b$ and t be an integer with $t \geq 2$. We call an integer k representable if there exist non-negative integers u, v with $ua + vb = k$. Suppose we have integers $l_1 < l_2 < \dots < l_t$ fulfilling*

- (1) $a < \widehat{w}_i = ab - l_i - 1 < ab - 1$, l_i not representable, $l_i + 1$ representable for $1 \leq i \leq 3$,
- (2) $0 < \sum_{j=1}^z l_{i_j} - (z - 1)ab < ab$ is not representable for all $2 \leq z < t$ and all subsets $\{i_1, \dots, i_z\} \subseteq \{1, \dots, t\}$ of cardinality z , and
- (3) $0 < \sum_{j=1}^t l_j + 1 - (t - 1)ab < ab$ is representable.

The weighted game $(ab; \widehat{w}_1, \widehat{w}_2 + 1, \dots, \widehat{w}_t + 1, a, \dots, a, b, \dots, b)$, $\tilde{n} = (1 \dots 1 \ b \ a)$ has the following t minimum sum representations preserving types:

- $(ab; \widehat{w}_1, \widehat{w}_2 + 1, \dots, \widehat{w}_t + 1, a, \dots, a, b, \dots, b)$
- $(ab; \widehat{w}_1 + 1, \widehat{w}_2, \widehat{w}_3 + 1, \dots, \widehat{w}_t + 1, a, \dots, a, b, \dots, b)$
- \vdots
- $(ab; \widehat{w}_1 + 1, \dots, \widehat{w}_{t-1} + 1, \widehat{w}_t, a, \dots, a, b, \dots, b)$

For $t \geq 4$ we have the following examples:

$t = 4$, $a = 19$, $b = 11$, $l_1 = 141$, $l_2 = 157$, $l_3 = 160$, and $l_4 = 179$.

$t = 5$, $a = 19$, $b = 17$, $l_1 = 249$, $l_2 = 251$, $l_3 = 253$, $l_4 = 268$, and $l_5 = 287$.

$t = 6$, $a = 29$, $b = 17$, $l_1 = 389$, $l_2 = 396$, $l_3 = 401$, $l_4 = 418$, $l_5 = 430$, and $l_6 = 447$.

$t = 7$, $a = 31$, $b = 29$, $l_1 = 746$, $l_2 = 750$, $l_3 = 752$, $l_4 = 777$, $l_5 = 779$, $l_6 = 808$, and $l_7 = 810$.

$t = 8$, $a = 37$, $b = 29$, $l_1 = 883$, $l_2 = 891$, $l_3 = 920$, $l_4 = 941$, $l_5 = 949$, $l_6 = 970$, $l_7 = 978$, and $l_8 = 1007$.

$t = 9$, $a = 41$, $b = 31$, $l_1 = 1086$, $l_2 = 1100$, $l_3 = 1106$, $l_4 = 1117$, $l_5 = 1127$, $l_6 = 1137$, $l_7 = 1158$, $l_8 = 1168$, and $l_9 = 1199$.

$t = 10$, $a = 43$, $b = 41$, $l_1 = 1513$, $l_2 = 1550$, $l_3 = 1552$, $l_4 = 1554$, $l_5 = 1593$, $l_6 = 1595$, $l_7 = 1597$, $l_8 = 1636$, $l_9 = 1638$, and $l_{10} = 1679$.

Being a bit careless one might conjecture that for a given t every weighted games can have at most $t - 2$ different minimum sum representations preserving types and that this bound is tight.

3.3. Possible weights of minimum integer representations

Instead of asking which classes of weighted games admit a unique minimum integer representation or a unique integer representation preserving types one can ask which weights are possible in a minimum integer representation. The following theorem and remarks completely resolve this question for two different weights except exact lower bounds on the number of necessary voters n .

Theorem 3.3. For two coprime integers $b > a \geq 1$ or $b = 1$, $a = 0$ the weighted game $[q = ab; \underbrace{b, \dots, b}_{n_1}, \underbrace{a, \dots, a}_{n_2}]$, where $n_1 \geq a + 1(2a + 1)$ and $n_2 \geq b + 1(2b + 1)$, is in minimum weighted representation.

PROOF. Let two integers u, v be uniquely defined via $ua + vb = q - 1$, where $0 \leq u \leq b - 1$ and $0 \leq v \leq a - 1$. Let us assume that $[q'; \underbrace{b', \dots, b'}_{n_1}, \underbrace{a', \dots, a'}_{n_2}]$ is a representation of the same game. Due to

$ua + vb < q$, $ab \geq q$, and $ba \geq q$ we have $ua' + vb' \leq q' - 1$, $ab' \geq q'$, and $ba' \geq q'$. Multiplying the first inequality by ab yields

$$abua' + abvb' \leq \underbrace{ab}_{=q} q' - \underbrace{ab}_{=q}.$$

Adding bv times the second inequality and au times the third inequality yields

$$abua' + abvb' \geq \underbrace{(au + bv)}_{=q-1} q'.$$

Thus we conclude $qq' - q' \leq qq' - q$, which is equivalent to $q' \geq q$. Next we deduce $b' \geq b$ and $a' \geq a$ from $ab' \geq q' \geq q = ab$ and $ba' \geq q' \geq q = ab$.

Now let us assume that $[q'; b'_1, \dots, b'_{n_1}, a'_1, \dots, a'_{n_2}]$ represents the same game, where $b'_1 \geq \dots \geq b'_{n_1} > a'_1 \geq \dots \geq a'_{n_2}$ are non-negative integers. Due to $ua + vb < q$, $ab \geq q$, and $ba \geq q$ we have

$$\begin{aligned} \sum_{i=1}^u a'_i + \sum_{i=1}^v b'_i &\leq q' - 1, \\ \sum_{i=0}^{a-1} b'_{n_1-i} &\geq q', \text{ and} \\ \sum_{i=0}^{b-1} a'_{n_2-i} &\geq q'. \end{aligned}$$

As before we conclude

$$ab \sum_{i=1}^u a'_i + ab \sum_{i=1}^v b'_i \leq qq' - q$$

and

$$au \sum_{i=0}^{b-1} a'_{n_2-i} + bv \sum_{i=0}^{a-1} b'_{n_1-i} \geq qq' - q'.$$

Due to $a'_i \geq a_{n_2-j}$ for all $1 \leq i \leq u$, $0 \leq j < b$ and $b'_i \geq b_{n_1-j}$ for all $1 \leq i \leq v$, $0 \leq j < a$ we can conclude $q' \geq q$. From $\sum_{i=0}^{b-1} a'_{n_2-i} \geq q' \geq q = ab$ we obtain $a'_{n_2-b+1} \geq a$, where equality holds if and only if $a'_{n_2-b+1} = \dots = a'_{n_2} = a$. So we have $a'_i \geq a$ for all $1 \leq i \leq n_2$ in the case of equality. Otherwise we have

$$\sum_{i=1}^{n_2} a'_i = \sum_{i=1}^{n_2-b} \underbrace{a'_i}_{\geq a+1} + \sum_{i=0}^{b-1} \underbrace{a'_{n_2-i}}_{\geq ab} \geq (n_2 - b)(a + 1) + ab \underbrace{>}_{b < n_2} n_2 a,$$

which would contradict minimality. Similarly we deduce $b'_i \geq b$ for all $1 \leq i \leq n_1$. \square

Remark 3.4.

- (1) If $r = \gcd(a, b) > 1$, then $[\frac{q}{r}; \frac{b}{r}, \dots, \frac{b}{r}, \frac{a}{r}, \dots, \frac{a}{r}]$ is a smaller representation for the same game.
- (2) If $b = a$ then there is only one type of voters with minimum representation $[q'; 1, \dots, 1]$ for a suitable quota q' . If $b < a$ then the voters of type 2 would be more powerful than the voters of type 1, which is not possible by definition.
- (3) If $a = 0$ and $b > 1$ then $[\lceil \frac{q}{b} \rceil, 1, \dots, 1, 0, \dots, 0]$ is a smaller representation for the same game.

(4) The lower bounds on n_1 and n_2 can be improved, e. g. based on the knowledge of u and v .

There is a generalization to weighted games with more than two types of voters:

Theorem 3.5. Let a_1, \dots, a_t be integers such that $a_1 > a_2 > \dots > a_t > 0$ and for each $1 \leq i \leq t$ there is an index $1 \leq j \leq t$ with $\gcd(a_i, a_j) = 1$. The weighted game

$$[q = \text{lcm}(a_1, \dots, a_t); \underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_t, \dots, a_t}_{n_t}],$$

where $n_i \geq 1 + 2 \max_j a_j$ for all $1 \leq i \leq t$, is in minimum weighted representation.

PROOF. For each index $1 \leq i \leq t$ choose a suitable index $j \neq i$ such that $\gcd(a_i, a_j) = 1$. Let z be the integer defined via $q = a_i a_j z$ and two integers u, v be uniquely defined via $u a_i + v a_j = a_i a_j - 1$, where $0 \leq u \leq a_j - 1$ and $0 \leq v \leq a_i - 1$. With this, new weights a'_1, \dots, a'_t , and a new quota q' we have

$$\begin{aligned} -((z-1)a_j + u)a'_i - v a'_j + q' &\geq 1 \\ (z-1)a_j a'_i + a_i a'_j - q' &\geq 0 \\ z a_j a'_i - q' &\geq 0. \end{aligned}$$

Combining these inequalities with the vectors $(a_i \quad v \quad a_i - v)$, $(a_j \quad a_j - u \quad u)$, and $(z a_i a_j \quad z a_j v \quad (z-1)a_j(a_i - v) + a_i u)$ yields $a'_i \geq a_i$, $a'_j \geq a_j$, and $q' \geq q$.

Analogously to the proof of the previous theorem we can treat the case of different weights within equivalence classes of voters. \square

In the next Theorem we pay less restrictive conditions on the weights a_i by a quite large bound on the number of voters n .

Theorem 3.6. For integers $a_1 > a_2 > \dots > a_t > 0$ with $\gcd(a_1, \dots, a_t) = 1$ the weighted game

$$[q = \prod_{j=1}^t a_j; \underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_t, \dots, a_t}_{n_t}],$$

where $n_i \geq 1 + 2 \prod_{j=1, j \neq i}^t a_j$ for all $1 \leq i \leq t$, is in minimum weighted representation.

PROOF. Let t integers u_i be uniquely defined via $\sum_{i=1}^t u_i a_i = q - 1$, where $0 \leq u_i \leq \prod_{j=1, j \neq i}^t a_j - 1$ for all $1 \leq i \leq t$. For new weights a'_1, \dots, a'_t and a new quota q' the following inequalities have to be valid:

$$\begin{aligned} \frac{q}{a_1} \cdot a'_1 - q' &\geq 0 \\ &\vdots \\ \frac{q}{a_t} \cdot a'_t - q' &\geq 0 \\ -\sum_{i=1}^t u_i a'_i + q' &\geq 1 \end{aligned}$$

Summing up $a_i u_i$ times the i th inequality plus q times the last inequality yields $q' \geq q$. Using this at the i th inequality gives $a'_i \geq a_i$ for all $1 \leq i \leq t$.

Analogously to the proofs of the previous theorems we can treat the case of different weights within equivalence classes of voters. \square

Obviously the condition $\gcd(a_1, \dots, a_t) = 1$ is necessary. If we also want to use zero weights we can utilize the next lemma:

Lemma 3.7. *The weighted game $[q; \underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_t, \dots, a_t}_{n_t}]$, where $a_i > 0$ for all $1 \leq i \leq t$, is in minimum weighted representation if and only if the weighted game $[q; \underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_t, \dots, a_t}_{n_t}, \underbrace{0, \dots, 0}_{n_{t+1}}]$ is in minimum weighted representation.*

4. Weighted voting games with minimum integer representations for two types of voters

As we have remarked in the introduction, all complete simple games with just one type of voters, $t = 1$, are weighted and admit a unique minimum integer representation with all weights being equal to 1. These games are called symmetric, anonymous or k-out-of-n-games in the literature. In the previous section the allowable values for "t" for possible minimum integer representations for weighted games has been narrowed, since for $t = 3$ there are weighted games without a unique minimum integer representation. So the central question of this section (and the paper) is: what occurs for two types of voters. We first state the main result and immediately after we prove it.

Theorem 4.1. *Each weighted game with two types of voters admits a unique minimum integer representation, where $w_1 \leq \max(n_1 + 1, n_2)$, $w_2 \leq \max(n_1, n_2 - 1)$, and $q \leq (n_1 + n_2) \cdot \max(n_1 + 1, n_2)$. For $r \geq 2$ the bounds can be sharpened a bit to $1 \leq w_1 \leq n_2$, $0 \leq w_2 \leq n_1$, and $w_2 + 1 \leq q \leq 2n_1 n_2$.*

4.1. Proof of the main theorem for $r = 1$

Capturing the different compact descriptions of complete simple games with $t = 2$ for $r = 1$ and $r \geq 2$ via linear inequalities, we start with the case of one shift-minimal winning coalition.

Theorem 4.2. *For a weighted game (n_1, n_2) , $((m_1, m_2))$ with two types of voters, i. e. $t = 2$ and $r = 1$, there exist a unique minimum integer representation.*

PROOF. Since the game is weighted there are some restrictions on the parameters m_1, m_2 , so that we utilize case differentiation. Let the weights be given by a_1, \dots, a_{n_1} and b_1, \dots, b_{n_2} . We have $a_i \geq b_j + 1$ for all $1 \leq i \leq n_1, 1 \leq j \leq n_2$.

At first, we determine those complete simple games which are not weighted. For $1 \leq m_1 \leq n_1 - 1$ and $2 \leq m_2 \leq n_2 - 2$ both coalitions $(m_1 - 1, m_2 + 2)$ and $(m_1 + 1, m_2 - 2)$ are losing coalitions so that we conclude $a_i > b_j + b_h$ and $a_i < b_j + b_h$ for all $1 \leq i \leq n_1$ and $1 \leq j < h \leq n_2$, which is a contradiction. The remaining complete simple games are weighted and we determine their unique minimum integer representations. In the following we also speak of losing vectors instead of losing coalitions.

- $1 \leq m_1 \leq n_1 - 1, m_2 = 0$:

We can easily check that $a_1 = \dots = a_{n_1} = 1, b_1 = \dots = b_{n_2} = 0, q = m_1$ represents the game and is the unique minimum integer representation.

- $1 \leq m_1 \leq n_1 - 1, m_2 = 1$:
Since $(m_1, 0)$ and $(m_1 - 1, n_2)$ are losing vectors, we can conclude $b_i \geq 1$ for all $1 \leq i \leq n_2$ and $a_i \geq n_2$ for all $1 \leq i \leq n_1$. We can easily check that $a_1 = \dots = a_{n_1} = n_2, b_1 = \dots = b_{n_2} = 1, q = m_1 n_2 + 1$ represents the game and thus is the unique minimum integer representation.
- $1 \leq m_1 \leq n_1 - 1, m_2 = n_2 - 1 \geq 2$:
If $m_1 + n_2 - 1 \leq n_1$ the shift-maximal losing coalitions are given by $(m_1 + n_2 - 2, 0)$ and $(m_1 - 1, n_2)$. Comparing the first shift-minimal losing coalition with the shift-minimal winning coalition and inserting $a_i \geq b_j + 1$ yields $b_j \geq n_2 - 1$ and $a_i \geq n_2$. We can easily check that $a_1 = \dots = a_{n_1} = n_2, b_1 = \dots = b_{n_2} = n_2 - 1, q = m_1 n_2 + (n_2 - 1)^2$ represents the game.

If $m_1 + n_2 - 1 > n_1$ the shift-maximal losing coalitions are given by $(n_1, m_1 + n_2 - 2 - n_1)$ and $(m_1 - 1, n_2)$. Comparing the first shift-maximal losing coalition with the shift-minimal winning coalition and inserting $a_i \geq b_j + 1$ yields $b_j \geq n_1 - m_1 + 1$ and $a_i \geq n_1 - m_1 + 2$. We can easily check that $a_1 = \dots = a_{n_1} = n_1 + 2 - m_1, b_1 = \dots = b_{n_2} = n_1 + 1 - m_1, q = (m_1 + n_2)(n_1 + 1 - m_1) + 2m_1 - n_1 - 1$ represents the game.
- $m_1 = n_1, 0 \leq m_2 \leq n_2 - 1$:
If $m_2 = 0$ then $a_1 = \dots = a_{n_1} = 1, b_1 = \dots = b_{n_2} = 0, q = n_1$ is the unique minimum representation. Otherwise the only shift-maximal losing coalitions are $(n_1, m_2 - 1)$ and $(n_1 - 1, n_2)$. From the winning coalition (n_1, m_2) and the losing coalition $(n_1, m_2 - 1)$ we deduce $b_i \geq 1$ for all $1 \leq i \leq n_2$. With this we deduce from the winning coalition (n_1, m_2) and the losing coalition $(n_1 - 1, n_2)$ for all $1 \leq i \leq n_1$ the inequality $a_i \geq \underbrace{n_2 - m_2}_{\geq 1} + 1$ and $q \geq n_1(n_2 - m_2 + 1) + m_2$. We can easily check that equality is possible, so that $a_i = n_2 - m_2 + 1, b_j = 1, q = n_1(n_2 - m_2 + 1) + m_2$ is the unique minimum representation.

□

4.2. Proof of the main theorem for $r > 1$

To circumvent some case differentiations in the remaining part we now completely handle the cases where null voters, i. e. voters i such that $\chi(U) = \chi(U \cup \{i\})$ for all subsets $U \subseteq N$, occur.

Lemma 4.3. *For a weighted game with two types of voters, where one class consists of null voters, there exists a unique minimum representation.*

PROOF. In this case the game has only one shift-minimal winning coalition and we can apply the previous theorem. □

Lemma 4.4. *The weight of a null voter is 0 in every minimum integer representation of a weighted game and each non-null voter has a weight of at least 1 in every integer representation.*

PROOF. Replacing the weight to 0 decreases weights and keeps the representation valid. The only integral weight being smaller than 1 is zero, which would lead to a null voter. □

In order to prove that each weighted game with two types of voters admits a unique minimum integer representation, we provide a lemma that allows us to restrict our considerations onto the cases, where only two (possibly non-integral) weights are used.

Lemma 4.5. *For a given weighted game with two types of voters, $r \geq 2$, and without null voters, let \mathcal{W}^s be the set of shift-minimal winning coalitions, \mathcal{L}^s be the set of shift-maximal losing coalitions, and $w = (w_1, w_2)^T$ be a weight vector. If there exist integers $\widehat{w}_1, \widehat{w}_2, \widehat{q}$ such that $(\widehat{w}_1, \widehat{w}_2, \widehat{q})$ is an optimal solution of the following three linear programs, where $c \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and $[\widehat{q}; \widehat{w}_1, \dots, \widehat{w}_1, \widehat{w}_2, \dots, \widehat{w}_2]$ is a representation of the game, then this representation is a minimum weighted integer representation.*

$$\min c \cdot (w_1, w_2, q)^T$$

subject to

$$\begin{aligned} \tilde{m}^T w &\geq q & \forall \tilde{m} \in \mathcal{W}^s \\ \tilde{l}^T w &\leq q - 1 & \forall \tilde{l} \in \mathcal{L}^s \\ w_1 &\geq w_2 + 1 \\ w_2 &\geq 1 \\ w_1, w_2, q &\geq 0 \end{aligned}$$

PROOF. We compare the special integral representation $[\widehat{q}; \widehat{w}_1, \dots, \widehat{w}_1, \widehat{w}_2, \dots, \widehat{w}_2]$, where \widehat{w}_1 , \widehat{w}_2 , and \widehat{q} are minimizers of three linear programs (without integrality conditions), with other integral representations. Therefore let us assume that $[q'; a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2}]$ is another integral representation of the weighted game. W.l.o.g. we assume $a_1 \geq \dots \geq a_{n_1} > b_1 \geq \dots \geq b_{n_2} \geq 1$. By \overline{w}_1 we denote the arithmetic mean of the a_i and by \overline{w}_2 the arithmetic mean of the b_i . We have $\tilde{m}^T \overline{w} \geq q'$ for all $\tilde{m} \in \mathcal{W}^s$, $\tilde{l}^T \overline{w} \leq q - 1$ for all $\tilde{l} \in \mathcal{L}^s$, $\overline{w}_1 \geq \overline{w}_2 + 1$, $\overline{w}_2 \geq 1$, and $\overline{w}_1, \overline{w}_2, q' \geq 0$. By considering the minimization of \overline{w}_1 , \overline{w}_2 , and q' (in all three linear programs) we conclude $\overline{w}_1 \geq \widehat{w}_1$, $\overline{w}_2 \geq \widehat{w}_2$, $q' \geq \widehat{q}$ and can assume that equality holds (otherwise the assumed integral representation is component-wise larger or equal to our special integer representation).

Since for each feasible linear program there exists at least one vertex, where the optimal solution is attained, we conclude: There have to be at least three inequalities in the linear programs, where the solution $(\overline{w}_1, \overline{w}_2, q')$ leads to equality. We call those inequalities tight inequalities.

Since we assume $\overline{w}_1 = \widehat{w}_1$, $\overline{w}_2 = \widehat{w}_2$, and $q' = \widehat{q}$, the arithmetic means and the quota q' have to be integers. Due to $\overline{w}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} a_i$ either we have $a_1 = \dots = a_{n_1} = \overline{w}_1$ or $a_1 \geq \overline{w}_1 + 1$ and $a_{n_1} \leq \overline{w}_1 - 1$. Analogously we have either $b_1 = \dots = b_{n_2} = \overline{w}_2$ or $b_1 \geq \overline{w}_2 + 1$ and $b_{n_2} \leq \overline{w}_2 - 1$.

So let us assume $a_1 \geq \overline{w}_1 + 1$ and $a_{n_1} \leq \overline{w}_1 - 1$. If $\tilde{l}^T w \leq q - 1$ is a tight inequality for $\tilde{l}^T = (l_1, l_2) \in \mathcal{L}^s$ then we must have $l_1 = n_1$ since otherwise the l_1 largest a_i would violate the inequality. In such a case we have $|\mathcal{W}^s| = 1$, which contradicts $r \geq 2$. Thus we can assume that no such inequality is tight. Since $\overline{w}_1 \geq 2$ and $\overline{w}_2, q' \geq 1$, there is at least one tight inequality of type $\tilde{m}^T w \geq q$ for $\tilde{m}^T = (m_1, m_2) \in \mathcal{W}^s$. Thus we have $m_1 = n_1$ since otherwise the m_1 smallest a_i would violate the inequality. This also means that there is exactly one inequality of type $\tilde{m}^T w \geq q$, which is tight, and that we have $\overline{w}_2 = 1$, $\overline{w}_1 = \overline{w}_2 + 1 = 2$. This would be a contradiction to $a_{n_1} > b_1$.

Now let us assume $b_1 \geq \overline{w}_2 + 1$ and $b_{n_2} \leq \overline{w}_2 - 1$. If $\tilde{m}^T w \geq q$ is a tight inequality for $\tilde{m}^T = (m_1, m_2) \in \mathcal{W}^s$ then we must have $m_2 = n_2$ since otherwise the m_2 smallest b_i would violate the inequality. If $\tilde{l}^T w \leq q - 1$ is a tight inequality for $\tilde{l}^T = (l_1, l_2) \in \mathcal{L}^s$ then we must have $l_2 = n_2$ since otherwise the l_2 largest b_i would violate the inequality. Both cases, that $\tilde{m}^T w \geq q$ and $\tilde{l}^T w \leq q - 1$ are tight, cannot occur simultaneously since otherwise we could conclude $\overline{w}_1 \leq 1$, which is not possible. Additionally there can be at most one such tight inequality. Thus the inequalities $w_1 \geq w_2 + 1$ and $w_2 \geq 1$ must be tight so that we have $\overline{w}_2 = 1$ and $\overline{w}_1 = \overline{w}_2 + 1 = 2$, which contradicts $a_{n_1} > b_1$.

Thus finally we can conclude $a_1, \dots, a_{n_1} \geq \overline{w_1} = \widehat{w_1}$, $b_1, \dots, b_{n_2} \geq \overline{w_2} = \widehat{w_2}$, and $q' \geq \widehat{q}$. \square

Having this lemma at hand it remains to prove that the three linear programs, without integrality conditions and consisting of three variables, from Lemma 4.5, have a common integral optimum solution for the three stated target functions. This means that there is a unique integral solution which minimizes the target functions $c_1 w_1 + c_2 w_2 + c_3 q$ for all $c_1, c_2, c_3 \geq 0$, where $c_1 + c_2 + c_3 > 0$. At first we show that we can drop the constraints $w_1 \geq w_2 + 1$ and $w_2 \geq 1$ without enlarging the feasible set of solutions.

Lemma 4.6. *The inequality $w_1 \geq w_2 + 1$ is redundant in Lemma 4.5 without $w_2 \geq 1$.*

PROOF. Let $(a, b) \in \mathcal{W}^s$ with minimal a . If $a \geq 1$ and $b < n_2$, then $(a - 1, n_2) \in \mathcal{L}^s$. With this we conclude

$$aw_1 + bw_2 \geq q \geq (a - 1)w_1 + n_2 w_2 + 1,$$

which is equivalent to $w_1 \geq \underbrace{(n_2 - b)}_{\geq 1} w_2 + 1$.

If $a = 0$ or $b = n_2$ then let $(c, d) \in \mathcal{W}^s$ with minimal $c > a$ (we assume $|\mathcal{W}^s| = r \geq 2$). With this we have $(c - 1, a + b - c) \in \mathcal{L}^s$ and conclude

$$cw_1 + dw_2 \geq q \geq (c - 1)w_1 + (a + b - c)w_2 + 1,$$

which is equivalent to $w_1 \geq \underbrace{(a + b - c - d)}_{\geq 1} w_2 + 1$. \square

Lemma 4.7. *For $|\mathcal{W}^s| = r \geq 2$ the inequality $w_2 \geq 1$ is redundant in Lemma 4.5.*

PROOF. Let $(a, b) \in \mathcal{W}^s$ with minimal a . Due to $|\mathcal{W}^s| = r \geq 2$ we have $b > 0$ and $(a, b - 1) \in \mathcal{L}^s$. Thus we have $aw_1 + bw_2 \geq q \geq aw_1 + (b - 1)w_2 + 1$, from which we conclude $w_2 \geq 1$. \square

So each optimal vertex of the linear program in Lemma 4.5 is determined by three tight inequalities of one of the types $\tilde{m}^T w \geq q$ or $\tilde{l}^T w \leq q - 1$, since $w_1, w_2, q \geq 0$ can not be attained with equality. In the following three lemmas we consider the possible cases.

Lemma 4.8. *Three tight inequalities of type $\tilde{m}^T w \geq q$ or three inequalities of type $\tilde{l}^T w \leq q - 1$ have to be either linearly dependent or do not determine a solution at all.*

PROOF. Consider the equation system

$$\begin{aligned} aw_1 + bw_2 &= z \\ cw_1 + dw_2 &= z \\ ew_1 + fw_2 &= z, \end{aligned}$$

where $z \in \{q, q - 1\}$. Eliminating z leaves

$$\begin{aligned} (a - c)w_1 + (b - d)w_2 &= 0 \\ (c - e)w_1 + (d - f)w_2 &= 0 \end{aligned}$$

which has either the unique solution $w_1 = w_2 = 0$, which is infeasible for the whole linear program, or an infinite number of solutions due to scaling. (In the latter case the equations are linearly dependent.) \square

Lemma 4.9. *Two tight inequalities of type $\tilde{m}^T w \geq q$ and one tight inequality of type $\tilde{l}^T w \leq q - 1$ lead to an integral solution $(\widehat{w}_1, \widehat{w}_2, \widehat{q})$ such that $w_1 \geq \widehat{w}_1$, $w_2 \geq \widehat{w}_2$, and $q \geq \widehat{q}$ for all feasible (w_1, w_2, q) or do not determine a solution at all.*

PROOF. Let $(a, b), (c, d) \in \mathcal{W}^s$ and $(e, f) \in \mathcal{L}^s$ be the tight coalitions where we assume $a > c$. Solving the corresponding equation system yields $\widehat{w}_1 = \frac{d-b}{Q}$, $\widehat{w}_2 = \frac{a-c}{Q}$, and $\widehat{q} = \frac{ad-bc}{Q}$, where $Q := fc - fa + ad - bc - ed + eb \in \mathbb{N}$.

Let $g := \gcd(a - c, d - b) \geq 1$. Since $(a - \frac{a-c}{g}, b + \frac{d-b}{g})$ is also a tight winning coalition we can assume $g = 1$.

Now we choose unique integers u, v fulfilling $u(d - b) - v(a - c) = 1$, where $0 < u \leq a - c$ and $0 \leq v < d - b$. The coalition $(a - u, b + v)$ has weight $q - \frac{1}{Q}$ and thus is losing. Since all losing coalitions have weight at most $q - 1$ we conclude $Q = 1$.

Let us have a closer look at the corresponding inequality system again:

$$\begin{aligned} aw_1 + bw_2 - q &\geq 0 \\ cw_1 + dw_2 - q &\geq 0 \\ -ew_1 - fw_2 + q &\geq 1 \end{aligned}$$

For the basis (w_1, w_2, q) the inverse matrix is given by

$$M^{-1} = \frac{1}{Q} \cdot \begin{pmatrix} d-f & f-b & d-b \\ e-c & a-e & a-c \\ ed-cf & af-eb & ad-bc \end{pmatrix}.$$

If we can show that all entries of M^{-1} are non-negative then we have $w_1 \geq \widehat{w}_1$, $w_2 \geq \widehat{w}_2$, and $q \geq \widehat{q}$ for all feasible (w_1, w_2, q) .

Since $Q = 1$ we can choose $e = a - u$, $f = b + v$. With this we have

$$\begin{aligned} d-f &\geq 1 \\ f-b &\geq 0 \\ d-b &\geq 2 \\ e-c &\geq 0 \\ a-e &\geq 1 \\ a-c &\geq 1 \\ ad-bc &\geq \underbrace{d-b}_{\widehat{q} > \widehat{w}_1} + 1 \geq 3 \\ ed-cf &\stackrel{Q=1}{=} ad-bc - (af-eb) - 1 = \underbrace{a(d-f)}_{\geq 1} + \underbrace{b(e-c)}_{\geq 0} - 1 \geq 0 \\ af-eb &= a(b+v) - (a-u)b = av + bu \geq 0 \end{aligned}$$

□

Let us have an example to illustrate how the lemma works. Therefore let $(4 \quad 8), \begin{pmatrix} 4 & 0 \\ 3 & 1 \\ 2 & 4 \\ 1 & 6 \\ 0 & 8 \end{pmatrix}$ be a weighted

game $(w_1 = 7, w_2 = 3, q = 24)$. Let us assume the winning coalitions $(3, 1), (0, 8)$ and the losing coalitions

(1, 5) would be tight coalitions. The solution of the corresponding equation system is given by $w_1 = \frac{7}{2}$, $w_2 = \frac{3}{2}$, $q = 12$. Since $1 \cdot (d - b) - 2 \cdot (a - c) = 1$ the coalition (2, 3) is a losing coalition with weight 11.5.

If we replace (1, 5) by (2, 3) we obtain $w_1 = 7$, $w_2 = 3$, $q = 24$, and

$$M^{-1} = \begin{pmatrix} 5 & 2 & 7 \\ 2 & 1 & 3 \\ 16 & 7 & 24 \end{pmatrix}.$$

Lemma 4.10. *For $|\mathcal{W}^s| = r \geq 2$ one tight inequality of type $\tilde{m}^T w \geq q$ and two tight inequalities of type $\tilde{l}^T w \leq q - 1$ lead to an integral solution $(\widehat{w}_1, \widehat{w}_2, \widehat{q})$ such that $w_1 \geq \widehat{w}_1$, $w_2 \geq \widehat{w}_2$, and $q \geq \widehat{q}$ for all feasible (w_1, w_2, q) or do not determine a solution at all.*

PROOF. Let $(a, b) \in \mathcal{W}^s$ and $(c, d), (e, f) \in \mathcal{L}^s$ be the tight coalitions where we assume $e > c$. Solving the corresponding equation system yields $\widehat{w}_1 = \frac{d-f}{Q}$, $\widehat{w}_2 = \frac{e-c}{Q}$, and $\widehat{q} = \frac{ad-fa+eb-bc}{Q}$, where $Q := fc - fa + ad - bc - ed + eb \in \mathbb{N}$.

Let $g := \gcd(e - c, d - f) \geq 1$. Since $(e - \frac{e-c}{g}, c + \frac{d-f}{g})$ is also a tight losing coalition we can assume $g = 1$.

Now we choose unique integers u, v fulfilling $u(d - f) - v(e - c) = 1$, where $0 < u \leq e - c$ and $0 \leq v < d - f$. The coalition $(c + u, d - v)$ has weight $q - 1 + \frac{1}{Q}$ and thus is winning. Since all winning coalitions have weight at least q we conclude $Q = 1$.

Let us have a closer look at the corresponding inequality system again:

$$\begin{aligned} aw_1 + bw_2 - q &\geq 0 \\ -cw_1 - dw_2 + q &\geq 1 \\ -ew_1 - fw_2 + q &\geq 1 \end{aligned}$$

For the basis (w_1, w_2, q) the inverse matrix is given by

$$M^{-1} = \frac{1}{Q} \cdot \begin{pmatrix} d-f & b-f & d-b \\ e-c & e-a & a-c \\ ed-cf & eb-af & ad-bc \end{pmatrix}.$$

If we can show that all entries of M^{-1} are non-negative then we have $w_1 \geq \widehat{w}_1$, $w_2 \geq \widehat{w}_2$, and $q \geq \widehat{q}$ for all feasible (w_1, w_2, q) .

Since $Q = 1$ we can choose $a = c + u$, $b = d - v$. With this we have

$$\begin{aligned} d-f &\geq 2 \\ b-f &\geq 1 \\ d-b &\geq 0 \\ e-c &\geq 1 \\ e-a &\geq 0 \\ a-c &\geq 1 \\ ed-cf &\geq \underbrace{c}_{e>c} \underbrace{(d-f)}_{\geq 2} \geq 2 \\ eb-af &\geq \underbrace{1}_{Q=1} + (ed-cf) - (ad-bc) = 1 + \underbrace{d(e-a)}_{\geq 0} + \underbrace{c(b-f)}_{\geq 1} \geq 1 \\ ad-bc &= ud + vc \geq 0 \end{aligned}$$

□

Theorem 4.11. *The linear program from Lemma 4.5 contains only integral vertices and has a unique solution for all target functions $c_1w_1 + c_2w_2 + c_3q$, where $c_1, c_2, c_3 \geq 0$ and $c_1 + c_2 + c_3 > 0$.*

PROOF. Combine the previous lemmas (the case $|\mathcal{W}^s| = r = 1$ is dealt directly). □

From the last theorem we can directly conclude our main theorem 4.1.

Remark 4.12. *Due to the above lemmas we can determine the unique minimum integer representation in $O(|\mathcal{W}^s|^3 \log(n_1 + n_2) \log \log(n_1 + n_2) + |\mathcal{W}^s|^2 \log^2(n_1 + n_2) \log \log(n_1 + n_2))$ time. Therefore we consider all pairs of shift-minimal winning coalitions and all pairs of shift-maximal losing coalitions ($|\mathcal{L}^s| \leq |\mathcal{W}^s| + 1$; $|\mathcal{W}| \leq \min(n_1 + 1, \lfloor \frac{n_2+2}{2} \rfloor) \leq \lfloor \frac{n+3}{3} \rfloor$ due to Inequalities (2)) and calculate the parameters u and v via the Euclidean algorithm to determine the third tight coalition. So we have to consider less than $|\mathcal{W}^s|^2 + |\mathcal{L}^s|^2$ cases. In each case the Euclidean algorithm performs at most $\log(n_1 + n_2)$ steps where numbers between $-n_1 - n_2$ and $n_1 + n_2$ are added and divided. After solving the 3×3 -equation system, which can be done in time $O(\log(n_1 + n_2) \log \log(n_1 + n_2))$, we only have to check if the solution is feasible. Checking the feasibility means determining the minimal weight of a winning coalition and the maximal weight of a losing coalition, which can be done using $O(|\mathcal{W}^s|)$ multiplications and additions.*

Since the minimal possible values of w_1 , w_2 , and q can be bounded via $w_1 \leq \max(n_1 + 1, n_2)$, $w_2 \leq \max(n_1, n_2 - 1)$, and $q \leq (n_1 + n_2) \cdot \max(n_1 + 1, n_2)$ we may also determine a minimal integer representation by trying out all possibilities, which results in a pseudo-polynomial algorithm.

Due to the famous LLL-algorithm [22, 23] integer linear programs with a fixed number of dimensions, i. e. the number of variables, and a fixed number of constraints can be solved in polynomial time. For a two variables integer program defined by m constraints involving coefficients with at most s bits there is a $O(m + \log m \log s)M(s)$ algorithm [6], where $M(s)$ is the time needed for s -bit integer multiplication (we assume $M(s) = s \log s \log \log s$). For $t = 2$ types of voters we have $|\mathcal{L}^s|, |\mathcal{W}^s| \leq \lfloor \frac{n+6}{3} \rfloor$, so that $m = |\mathcal{W}^s| \cdot |\mathcal{L}^s| + n \in O(n^2)$, and $s \in O(\log n)$ using the ILP formulation without the quota q . For a general but fixed number of variables Clarkson's sampling algorithm needs an expected number of $O(m + s \log m)$ arithmetic operations [5]. Using the ILP formulation with an extra variable for the quota q we have $m = |\mathcal{W}^s| + |\mathcal{L}^s| + n \in O(n^{t-1})$ and $s \in O(\log n)$ for t types of voters. We would like to remark that the number of shift-minimal winning coalitions can be exponential in n whenever the number t of types of voters is not restricted, see e. g. [19].

5. Enumerations and bounds for the number of weighted games

Besides from studying properties of complete simple games and weighted games one can also enumerate these special classes of cooperative games for small numbers of players n . In some cases enumeration results provide a deeper understanding. So far the number of complete simple games of weighted games is only known up to $n = 9$, see e. g. [10, 21]. Additionally restricting the parameters t (the number of types of voters) and/or r (the number of shift-minimal winning coalitions) opens the possibility to determine enumeration formulas in some cases. A widely known result in this context is $csg(n, 1) = wvg(n, 1) = n$, where $csg(n, t)$ denotes the number of complete simple games with n voters partitioned into t equivalence classes. Similarly $wvg(n, t)$ denotes the number of weighted games with n voters occurring in t different types. In [12] the authors have determined the formula $cs(n, 2) = Fib(n + 6) - (n^2 + 4n + 8)$, where $Fib(n)$ denotes the Fibonacci numbers, see also [21] for an alternative proof. So we know that $cs(n, t)$ is

at least exponential in n for $t \geq 2$. In this section we want to show that the situation changes for weighted games by proving a polynomial upper bound on $wm(n, t)$ in Theorem 5.2 and Theorem 5.3. It remains a task to come up with an exact formula for $wm(n, 2)$.

If we refine our counts to the numbers $csg(n, t, r)$ and $wvg(n, t, r)$ by additionally considering the number r of shift-minimal winning coalitions, more results can be obtained. In [21] an algorithm is given to principally determine an exact formula for $csg(n, t, r)$ whenever t and r are fixed. So far it is not known whether this can also be done for the number $wvg(n, t, r)$ of weighted games with t types of voters and r shift-minimal winning coalitions. For $r = 1$ it is not too difficult to come up with such enumeration formulas as we will demonstrate for $t = 2$. Having an exact characterization of the weighted games with $t = 2$ and $r = 1$ at hand, see the proof of Theorem 4.2, we can easily determine a formula for their number:

Corollary 5.1. *The number $wm(n, 2, 1)$ of weighted games with $t = 2$ and $r = 1$ is given by $n - 1$ for $n \leq 2$ and $2(n - 2)^2 + 2$ for $n \geq 3$.*

If we skip the parameter r then we can only state upper bound:

Theorem 5.2. $wm(n, 2) \leq \frac{n^5}{15} + 4n^4$.

PROOF. Due to the bounds in the minimum integer representation for $r \geq 2$ in Theorem 4.1 and Corollary 5.1 the number $wm(n, 2)$ of weighted games with n voters and two types of voters is upper bounded by

$$2(n - 2)^2 + 2 + \sum_{n_1=1}^{n-1} \sum_{w_1=1}^{n-n_1} \sum_{w_2=0}^{n_1} \sum_{q=1}^{2n_1(n-n_1)} 1 = 2(n - 2)^2 + 2 + 2 \sum_{n_1=1}^{n-1} (n - n_1)^2 (n_1 + 1) n_1 \leq \frac{n^5}{15} + 4n^4.$$

□

For an arbitrary number t of types of voters we can determine the following polynomial upper bound:

Theorem 5.3.

$$wm(n, t) < (tn)^{t^3+2t^2}.$$

PROOF. Let us denote the weight vector by w , the shift-minimal winning coalitions by m_i , and the shift-maximal losing coalitions by l_j . A complete simple game described by the m_i or l_j is weighted *if and only if* the system of inequalities

$$(m_i - l_j) w^T > 0 \quad (3)$$

(for all i, j) has a non-negative solution w .

Since λw is also a solution for all $\lambda > 0$ whenever w is a solution, we consider the equivalent system

$$(m_i - l_j) w^T \geq 1. \quad (4)$$

Such a system of linear inequalities corresponds to a polytope whose vertices correspond to n -element subsets of the constraints which are attained with equality. Using the fact that the coefficients of this system of linear inequalities are integers between $-(n - 1)$ and $n - 1$ we can apply Cramers rule to conclude that vertices of this polytope can be written as $v_i = (w_1 \ \dots \ w_t) = \left(\frac{a_{2,i}}{b_{2,i}} \ \dots \ \frac{a_{t,i}}{b_{t,i}} \right)$, where $0 \leq a_{j,i} \leq (t - 1)!(n - 1)^t$ and $1 \leq b_{j,i} \leq (t - 1)!(n - 1)^t$. Here the common denominator g is bounded from above by $\left((t - 1)!(n - 1)^t \right)^t$.

Thus multiplying vertex v_i with g yields integer weights \tilde{w}_i between 0 and $\left((t-1)!(n-1)^t\right)^{t+1}$. There are at most $(tn)^{t^3+t^2}$ possible tuples of integer weights to be considered. The quota can be chosen as the minimum weight of a winning coalition. Since there are less than n^t possibilities for the numbers n_i of voters in the t equivalence classes the proposed upper bound on $wm(n, t)$ follows. \square

6. Conclusion

The main result of this paper proves that weighted games with two types of players admit a unique minimum integer representation. On the contrary, for three types of players we generate examples without a unique integer representation. The addition of these two results to those already known allows us to conclude that a weighted game with k types of voters admits a minimum integer representation *if and only if* $k < 3$.

Concerning weighted representations preserving types we found examples of weighted games with 4 types of voters without a unique representation in integers. Nevertheless, it is still an open problem to elucidate whether weighted games with 3 types of voters admit a unique minimum representation in integers. To this end we have tried to generalize our technique from Subsection 4.2, i. e. we may consider the linear program minimizing the sum of the weights and have a closer look at the corners of the corresponding polytope, which are characterized by four equations corresponding to four *tight coalitions* (shift-maximal losing or shift-minimal winning coalitions).

As demonstrated in Subsection 4.2 for three tight coalitions, the resulting weights and the quota could be fractional. But using the extended Euclidean algorithm we were able to construct another coalition which contradicts the tightness of the starting three coalitions in these cases. For four tight coalitions (and the variables q, w_1, w_2 , and w_3) we may go along the same lines and use the extended Euclidean algorithm for three integers in order to deduce some restrictions on quadruples of tight coalitions. This indeed works, but there still remain cases where the optimal LP solution is fractional. By generating random weighted games with three types of voters we have discovered several such examples, some of them are given below. For each example we state the sizes of the equivalence classes $\tilde{n} = (n_1 \ n_2 \ n_3)$, the minimum non-integral weights $\bar{w} = (w_1 \ w_2 \ w_3 \ q)$, and the minimum integral weights $\hat{w} = (w_1 \ w_2 \ w_3 \ q)$:

- (1) $\tilde{n} = (9 \ 62 \ 71), \bar{w} = (38.\bar{3} \ 22.\bar{6} \ 6.\bar{6} \ 154.\bar{3}), \hat{w} = (46 \ 27 \ 8 \ 185)$
- (2) $\tilde{n} = (19 \ 52 \ 65), \bar{w} = (200 \ 110 \ 76.6 \ 3984.2), \hat{w} = (282 \ 155 \ 108 \ 5617)$
- (3) $\tilde{n} = (30 \ 93 \ 30), \bar{w} = (22.\bar{3} \ 16 \ 9.\bar{3} \ 122.\bar{3}), \hat{w} = (24 \ 17 \ 10 \ 131)$
- (4) $\tilde{n} = (8 \ 99 \ 10), \bar{w} = (17 \ 10.5 \ 4.5 \ 51), \hat{w} = (19 \ 12 \ 5 \ 57)$
- (5) $\tilde{n} = (3 \ 71 \ 37), \bar{w} = (100 \ 31.5 \ 15 \ 347.5), \hat{w} = (127 \ 40 \ 19 \ 441)$

Originally we have obtained the values of \hat{w} by minimizing $w_1 + w_2 + w_3$ but it turned out that in all of these (and the other found) cases we have a unique minimum integer representation preserving types, so that minimizing w_1, w_2, w_3 , or q would yield the same result. We would like to remark that we have also found some example where only one value is non-integral. Although, in our experiments the only occurring denominators were 2, 3, and 5, we do not think that the denominator is bounded by a constant. So far we have a very poor probabilistic model which generates those examples with a very low probability. Nevertheless we have a strong feeling that each weighted game with three types of voters admits a unique minimum integer representation preserving types. As a small justification we would like to remark that we have tried some specific parametric constructions which provable do not contain counter examples.

So at this point we leave this challenging question open for the interested reader and hope that our specific examples might help to get some useful insights. One can get a first glimpse of the difficulty of this problem by comparing the values of \overline{w} and \widehat{w} in our examples.

Weighted voting games with an arbitrary number of minimal integer representations have been generated. Moreover, some bounds have been obtained for the number of non-isomorphic weighted games depending on the number of voters and on the number of types of voters, and the existence of a weighted game, in minimum integer representation for any pair of two coprime integer weights, has been determined.

Other interesting open problems in the context of this paper are the question for a weighted game with a unique integer minimum sum representation, but without a minimum integer representation, and the question for a polynomial time algorithm to determine minimum sum integer representations for weighted games or a proof that this problem is *NP*-hard.

Another important line of research would be to study links between minimum representations of weighted games and some one-point solution concepts, like nucleolus, least core, etc., see e. g. [19, 27].

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